The Reason for the Efficiency of the Pian–Sumihara Basis

S. J. Childs*

Department of Pure and Applied Mathematics, Rhodes University, Grahamstown, 6140, South Africa

B. D. Reddy

Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch, 7700, South Africa

Abstract

A logical explanation as to why the choice of

$$\begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(the Pian–Sumihara basis) as a linear basis to approximate stress leads to greater efficiency in enhanced strain problems, is presented. An Airy stress function and the consequent selective simplification resulting from the differentiation of an implied, single, parent approximating polynomial, are the essence of this argument.

Keywords: Enhanced strain; Pian–Sumihara; Airy stress function; finite elements.

1 Introduction

Pian and Sumihara first identified the basis

$$\begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

as the most efficient linear basis for approximating stress in enhanced strain problems. This observation they made more rigorous by way of a Wilson element (a perturbation of sorts).

*Corresponding author. Email: *schilds@iafrica.com*

This paper presents a logical mathematical argument for making the same choice of basis, albeit with the wisdom of hindsight. It attributes the greater efficiency of the basis to properties inherent in the mathematics of the problem. The components of the stress tensor are recognised to be related by way of an Airy stress function and it is in this way that a fundamentally more correct representation of the full linear basis is arrived at. By further desiring the advantages of a two field problem, the most efficient, linear basis is obtained.

2 An Airy Stress Function

The Airy stress function is a potential of sorts. Interpreting stresses to be the various second derivatives of a single polynomial leads to selective simplification and interdependence between the resulting linear approximations. This simplification and the interdependence are not obvious in a more superficial treatment.

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{0} \qquad \Rightarrow \qquad \frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\sigma_{12}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial\sigma_{21}}{\partial x} + \frac{\partial\sigma_{22}}{\partial y} = 0$$

This is recogniseable as

$$\operatorname{curl}(-\sigma_{12}, \sigma_{11}, 0) = 0 \quad \text{and} \quad \operatorname{curl}(\sigma_{22}, -\sigma_{21}, 0) = 0.$$

This, in turn, implies that $(-\sigma_{12}, \sigma_{11}, 0)$ and $(\sigma_{22}, -\sigma_{21}, 0)$ may be interpretted as $\nabla\alpha$ and $\nabla\beta$ respectively, without any inconsistancy in the

$$\operatorname{curl}\nabla(\,\cdot\,) = 0$$

identity.

By symmetry of $\boldsymbol{\sigma}$,

$$\sigma_{12} = \sigma_{21} \Rightarrow \frac{\partial\alpha}{\partial x} - \frac{\partial\beta}{\partial y} = 0$$

and for a two dimensional problem of the type under consideration this once again implies

$$\operatorname{curl}(\beta, \alpha, 0) = 0.$$

$(\beta, \alpha, 0)$ may therefore be interpretted as $\nabla\Phi$ without any inconsistancy in the

$$\operatorname{curl}\nabla(\,\cdot\,) = 0$$

identity.

In summary, with an equation

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$$

governing the motion, in the two-dimensional case, the components of the stress may be derived from an Airy stress function as follows

$$\sigma_{11} = \frac{\partial^2\Phi}{\partial y^2},$$

$$\begin{aligned}\sigma_{22} &= \frac{\partial^2\Phi}{\partial x^2},\\ \sigma_{12} &= \frac{\partial^2\Phi}{\partial x\partial y},\end{aligned}$$

where Φ is the Airy stress function.

2.1 Finite Element Approximation

Due to approximation,

$$\mathrm{div}\,\boldsymbol{\sigma} = \mathbf{0}$$

and not the constitutive

$$\mathrm{div}\,\boldsymbol{\sigma} = \boldsymbol{f}$$

are really the equations being solved (Reddy [2]).

Defining a function

$$\phi(\xi,\eta) \equiv \Phi(x(\xi,\eta), y(\xi,\eta))$$

on each element Ω_e,

$$\begin{aligned}\sigma_{22} &= \frac{\partial^2\Phi}{\partial x_1^2}\\ &= \frac{\partial}{\partial\xi}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial x_1} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x_1}\right)\frac{\partial\xi}{\partial x_1} + \frac{\partial}{\partial\eta}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial x_1} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x_1}\right)\frac{\partial\eta}{\partial x_1}\\ &= \left(\frac{\partial^2\phi}{\partial\xi^2}\frac{\partial\xi}{\partial x_1} + \frac{\partial^2\phi}{\partial\eta\partial\xi}\frac{\partial\eta}{\partial x_1}\right)\frac{\partial\xi}{\partial x_1} + \left(\frac{\partial^2\phi}{\partial\eta\partial\xi}\frac{\partial\xi}{\partial x_1} + \frac{\partial^2\phi}{\partial\eta^2}\frac{\partial\eta}{\partial x_1}\right)\frac{\partial\eta}{\partial x_1}\end{aligned}$$

Assumption

The individual elements, Ω_e, are usually mapped to the master element, $\hat{\Omega}$, with $\frac{\partial\xi}{\partial x_2} \approx \frac{\partial\eta}{\partial x_1} \approx 0$ on average, $\frac{\partial\xi}{\partial x_1} \approx a_1$ and $\frac{\partial\eta}{\partial x_2} \approx a_2$, a_1 and a_2 some constants, on average. (Alternatively it can be argued that there will be no loss of generality or weakening of the argument if a rectangular mesh is considered. Not allowing this simplification leads to an extremely messy argument, a chapters long exercise in differentiation.) This implies

$$\sigma_{22} = a_1^2\frac{\partial^2\phi}{\partial\xi^2}.$$

Similarly,

$$\begin{aligned}\sigma_{11} &= a_2^2\frac{\partial^2\phi}{\partial\eta^2}\\ \sigma_{12} &= \sigma_{21} = a_1 a_2\frac{\partial^2\phi}{\partial\xi\partial\eta}\end{aligned}$$

3 The Relationship Implicit in the Linear Approximation

Since linear approximations of σ_{11} are to be considered,

$$\frac{\partial^2\phi}{\partial\eta^2} = b_1 + b_2\xi + b_3\eta$$

where b_1, b_2 and b_3 are the relevant combining constants. This means

$$\begin{aligned}\phi(\xi,\eta) &= \int_{-1}^{1}\int_{-1}^{1} b_1 + b_2\xi + b_3\eta \; d\eta d\eta \\ &= c_1 + c_3\eta + \frac{1}{2}b_1\eta^2 + \frac{1}{2}b_2\xi\eta^2 + \frac{1}{6}b_3\eta^3 + \eta f_1(\xi) + f_2(\xi) \qquad (1)\end{aligned}$$

in which the exact form of $\eta f_1(\eta) + f_2(\eta)$ remains to be determined. Similarly, approximating $\sigma_2 2$ as some multiple of $b_4 + b_5\xi + b_6\eta$ implies this very same polynomial function

$$\begin{aligned}\phi(\xi,\eta) &= \int_{-1}^{1}\int_{-1}^{1} b_4 + b_5\xi + b_6\eta \; d\xi d\xi \qquad \textit{(by Airy stress function)} \\ &= c_1 + c_2\xi + \frac{1}{2}b_4\xi^2 + \frac{1}{6}b_5\xi^3 + \frac{1}{2}b_6\xi^2\eta + \xi g_1(\eta) + g_2(\eta), \qquad (2)\end{aligned}$$

in which the exact form of $g_2(\eta)$ is determined by equation (1). This equation in turn specifies $f_2(\xi)$ in equation (1). Approximating $\sigma_{12} = \sigma_{21}$ in it's turn as as $b_7 + b_8\xi + b_9\eta$ implies the polynomial function

$$\begin{aligned}\phi(\xi,\eta) &= \int_{-1}^{1}\int_{-1}^{1} b_7 + b_8\xi + b_9\eta \; d\xi d\eta \\ &= c_1 + b_7\xi\eta + \frac{1}{2}b_8\xi^2\eta + \frac{1}{2}b_9\xi\eta^2 + f_2(\xi) + g_2(\eta) \qquad (3)\end{aligned}$$

where $f_2(\xi)$ and $g_2(\eta)$ have already been determined by equations (2) and (1) respectively. This last expression for $\phi(\xi,\eta)$ also specifies the, until now undetermined, $\eta f_1(\xi)$ and $\xi g_1(\eta)$ in equations (1) and (2). In summary, collecting equations (1), (2) and (3) together leads to the specification of an implied, single parent approximating polynomial

$$\phi(\xi,\eta) = c_1 + c_2\xi + c_3\eta + c_4\xi^2 + c_5\xi\eta + c_6\eta^2 + c_7\xi^3 + c_8\xi^2\eta + c_9\xi\eta^2 + c_{10}\eta^3.$$

Having established both the existance and nature of the relationship between the constants in what were apparently seperate linear approximations,

$$\begin{aligned}\frac{\partial^2\phi}{\partial\xi^2} &= 2c_4 + 6c_7\xi + 2c_8\eta \\ \frac{\partial^2\phi}{\partial\eta^2} &= 2c_6 + 2c_9\xi + 6c_{10}\eta \\ \frac{\partial^2\phi}{\partial\xi\partial\eta} &= c_5 + 2c_8\xi + 2c_9\eta\end{aligned}$$

can now be written where the c_i's $(i = 4, \cdots 10)$ are constants related to the finite element solution of the problem in question.

Conclusion

The Airy stress function therefore reveals how a linear approximation of the components of $\boldsymbol{\sigma}$ on each element really amounts to

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \eta & 0 & \xi & 0 \\ 0 & 1 & 0 & 0 & \xi & 0 & \eta \\ 0 & 0 & 1 & 0 & 0 & \eta & \xi \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad (4)$$

instead of the superficially more obvious

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \xi & 0 & 0 & \eta & 0 & 0 \\ 0 & 1 & 0 & 0 & \xi & 0 & 0 & \eta & 0 \\ 0 & 0 & 1 & 0 & 0 & \xi & 0 & 0 & \eta \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

4 Eliminating the Last Two Columns

The rank of the matrix in equation (4) indicates that there are still two extra columns. The equation in which $\boldsymbol{\sigma}$ is used is a three–field problem, in which the strain, $\boldsymbol{\gamma}$, only occurs once in a term $\boldsymbol{\sigma} \cdot \boldsymbol{\gamma}$. Choosing $\boldsymbol{\sigma}$ correctly would reduce the problem to a two–field problem since

$$\int \boldsymbol{\sigma} \cdot \boldsymbol{\gamma} d\boldsymbol{\Omega} \quad = \quad 0$$

is required in accordance with Reddy [3]. In other words

$$\boldsymbol{\sigma} \cdot \boldsymbol{\gamma} \quad = \quad \begin{bmatrix} 1 & 0 & 0 & \eta & 0 & \xi & 0 \\ 0 & 1 & 0 & 0 & \xi & 0 & \eta \\ 0 & 0 & 1 & 0 & 0 & \eta & \xi \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} \cdot \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

must always be zero. This is only certaint if the sixth and seventh columns of the stress basis are omitted.

5 Conclusion

An Airy stress function and consequent simplification resulting from the differentiation of an implied, single, parent, approximating polynomial are able to provide a logical explanation as to why the choice of

$$\begin{bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(the Pian–Sumihara basis) as a linear basis to approximate stress leads to greater efficiency in enhanced strain problems.

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